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On the choice of the reference temperature for fully-developed mixed convection in a vertical channel

A. Barletta*, E. Zanchini

Dipartimento di Ingegneria Energetica, Nucleare e del Controllo Ambientale (DIENCA), Università di Bologna, Viale Risorgimento, 2, I-40136 Bologna, Italy

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Abstract

The effect of the choice of the reference fluid temperature on the solutions of fully-developed mixed-convection problems in a plane vertical channel is studied. First, the boundary conditions of either uniform wall temperatures or a uniform temperature at a wall and a uniform heat flux on the opposite wall are considered. It is shown that, in these cases, the choice of the reference temperature affects both the velocity profiles and the axial change of the difference between the pressure and the hydrostatic pressure. A general method to choose the reference fluid temperature for the fully developed mixed convection in ducts is proposed. Finally, an analytical solution for the boundary condition of uniform wall heat fluxes is obtained by choosing the mean fluid temperature in each cross section as the local reference temperature. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Several studies on mixed-convection problems for a Newtonian fluid in a vertical channel have already been presented in the literature. In particular, some analytical solutions for the fully-developed flow have been performed. The boundary condition of uniform but different wall temperatures has been analysed by Aung and Worku [1]. The boundary conditions of uniform wall temperatures, of a uniform temperature on a wall and a uniform wall heat flux on the opposite wall, of uniform wall heat fluxes, have been studied by Hamadah and Wirtz [2] and by Cheng et al. [3]. The effect of viscous dissipation on the velocity and on the temperature profiles has been analysed by Barletta [4] for the boundary condition of uniform wall temperatures and by Zanchini [5] for boundary conditions of the third kind.

In this paper, the basic problems solved in Refs. [1–3] are reconsidered because, even if the mathematics developed in these papers is correct, the physical understanding of the problem cannot yet be considered as satisfactory. Moreover, the case of uniform wall heat fluxes with opposite signs has not yet been analysed. The defects in the physical interpretation are caused by the following circumstance. In Refs. [1–3], as well as in other theoretical papers on mixed convection, the reference temperature T_0, employed in the linear expression of the fluid density ρ as a function of temperature, is left undetermined. On the other hand,

* Corresponding author. Tel.: +39-051-6443295; fax: +39-051-6443296..

E-mail address: antonio.barletta@mail.ing.unibo.it (A. Barletta)

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Nomenclature

a_i	dimensionless coefficients, which appear in Eq. (61), $i=1, 2, 3, 4$
A, B	dimensionless integration constants, defined by Eq. (20)
c_p	specific heat at constant pressure [J kg^{-1} K^{-1}]
F	function of T_0, defined by Eq. (34) [K^2 m]
g	acceleration due to gravity [m s^{-2}]
Gr	Grashof number, defined in Eqs. (10) and (36)
k	thermal conductivity [W m^{-1} K^{-1}]
L	half the channel width [m]
Nu_1, Nu_2	Nusselt numbers at the channel walls, defined by Eq. (71)
p	pressure [Pa]
P	$=p+\rho_0 gX$, difference between the pressure and the hydrostatic pressure [Pa]
q_1, q_2	wall heat fluxes per unit area [W m^{-2}]
R	$=q_1/q_2$, dimensionless parameter
Re	Reynolds number, defined in Eq. (10)
T	temperature [K]
T_b	bulk temperature, defined by Eq. (72) [K]
T_m	mean fluid temperature, defined by Eqs. (8) and (38) [K]
T_0	reference fluid temperature [K]
T_1, T_2	wall temperatures [K]
u	$=U/U_m$, dimensionless X-component of the velocity
U	X-component of the velocity [m s^{-1}]
U_m	mean value of U, defined by Eq. (9) [m s^{-1}]
$\mathbf{U}$	velocity [m s^{-1}]
v	dimensionless X-component of the velocity, defined by Eq. (25)
X	axial coordinate [m]
y	$=Y/L$, dimensionless transverse coordinate
Y	transverse coordinate [m]

Greek symbols

β	thermal expansion coefficient [K^{-1}]
γ	dimensionless parameter, defined in Eqs. (10) and (36)
ε	dimensionless parameter, defined in Eq. (10)
η	auxiliary dimensionless variable, which appears in Eq. (59)
θ	dimensionless temperature, defined in Eqs. (10) and (36)
θ_b	dimensionless bulk temperature, defined by Eq. (74)
λ	dimensionless parameter, defined in Eq. (10)
$\tilde{\lambda}$	dimensionless parameter, defined by Eq. (30)
Λ	dimensionless parameter, defined by Eq. (24)
μ	dynamic viscosity [Pa s]
ρ	fluid density [kg m^{-3}]
ρ_m	fluid density at temperature T_m [kg m^{-3}]
ρ_0	reference fluid density [kg m^{-3}]
ω	dimensionless parameter, defined by Eq. (60)
ω_i	values of ω, roots of Eq. (65), $i=0, 1, 2, 3, 4$

an analysis of Refs. [1–3] suggests that the choice of T_0 may be important.

For instance, let us consider the boundary condition of uniform wall temperatures, hereafter denoted by $T_1 \leftrightarrow T_2$. For this boundary condition, profiles of the dimensionless axial velocity $u=U/U_m$, where U is the axial fluid velocity and U_m is the mean value of U in any cross section of the channel, are plotted in Refs. [1–3]. The plots represent u as a function of the dimensionless transverse coordinate and of two dimensionless parameters, each of which depends on T_0. Thus, it is natural to wonder whether the values of u obtained depend on the choice of the reference temperature T_0. In this paper, it will be proved that, even if slightly, these values depend on T_0. A more crucial problem concerns the values of the derivative dP/dX, where P is the difference between the pressure p and the hydrostatic pressure, and X is the axial coordinate. No result for this quantity is presented in Ref. [2]. According to the expressions reported in Refs. [1,3], which agree with each other and are confirmed by that obtained in this paper, the dependence of dP/dX on T_0 is very strong. As a consequence, dP/dX cannot be endowed with a clear physical meaning, unless a proper choice of T_0 is performed.

The comments on the choice of T_0 stated above hold also for the boundary condition of a uniform wall temperature and a uniform wall heat flux, hereafter denoted by $T_1 \leftrightarrow q_2$. Indeed, as it will be shown in Section 3, this boundary condition yields uniform wall temperatures and thus describes the same physical situation as the boundary condition $T_1 \leftrightarrow T_2$. For the boundary condition of uniform wall heat fluxes, hereafter denoted by $q_1 \leftrightarrow q_2$, the choice of the reference fluid temperature plays an even more crucial role. As it will be shown in Section 4, in this case any choice of a fixed reference fluid temperature, T_0, implies a quite unlikely pressure field.

Arpaci and Larsen [6] consider T_0 as an unknown quantity, and propose the following method to determine T_0 for one-dimensional parallel flows. The fluid motion is considered as the superposition of a forced flow, in which buoyancy forces vanish, and a buoyancy-driven flow, in which dP/dX vanishes. The solution of the buoyancy-driven problem, with the condition $dP/dX=0$, yields the value of T_0. The method can be applied only if the mass, momentum and energy balance equations are linear. For fully-developed channel flows, this condition holds if the viscous-dissipation effects are negligible and the boundary condition is either $T_1 \leftrightarrow T_2$ or $T_1 \leftrightarrow q_2$. In these cases, the method proposed in Ref. [6] yields the result $T_0=(T_1+T_2)/2$. Moreover, T_0 coincides with the mean fluid temperature T_m, because T is independent of X and is a linear function of the transverse coordinate. As it will be shown in Section 4, if the boundary condition is $q_1 \leftrightarrow q_2$ the method proposed by Arpaci and Larsen [6] cannot be applied, because the energy balance equation is not linear. A method to perform a suitable choice of the reference fluid temperature, for any fully developed channel flow, is proposed in this paper. For the boundary condition $T_1 \leftrightarrow T_2$ (or $T_1 \leftrightarrow q_2$) and negligible viscous-dissipation effects, this method agrees with that proposed in Ref. [6].

The paper is organized as follows. In Section 2, the boundary condition $T_1 \leftrightarrow q_2$ is considered and the effects of the choice of T_0 on u and on dP/dX are analysed. In particular, it is shown that the choice of the mean fluid temperature T_m as the reference temperature yields values of dP/dX which are unaffected by the buoyancy forces. On the other hand, the choices $T_0 \neq T_m$ imply that, in some flow conditions, P increases along the motion. These results suggest that only the choice $T_0=T_m$ gives to dP/dX an acceptable physical meaning. Finally, a more general argument in favour of the choice $T_0=T_m$ is discussed. In Section 3, it is shown that the solution obtained, in dimensionless form, for the boundary condition $T_1 \leftrightarrow q_2$, holds also for the boundary condition $T_1 \leftrightarrow T_2$, with proper definitions of the dimensionless parameters. In Section 4, the boundary condition $q_1 \leftrightarrow q_2$ is considered. First, it is shown that a reference fluid temperature variable with X must be chosen to obtain a reliable pressure field, and the choice $T_0(X)=T_m(X)$ is performed. Then, an analytical solution of the fully-developed mixed-convection problem in a vertical channel with the boundary condition $q_1 \leftrightarrow q_2$ is presented. This solution holds both for positive and for negative values of the ratio $R=q_1/q_2$, and includes the solution for the boundary conditions $T_1 \leftrightarrow q_2$ and $T_1 \leftrightarrow T_2$ as a special case, for $R=-1$.

2. Mixed convection problem with the boundary condition $T_1 \leftrightarrow q_2$: effect of the choice of the reference temperature

In this section, the effect of the choice of the reference temperature on the solution of the fully-developed mixed-convection problem in a vertical channel with the boundary condition $T_1 \leftrightarrow q_2$ is analysed.

Let us consider the steady and laminar flow of a Newtonian fluid in a parallel-plate vertical channel. The X-axis lies on the axial plane of the channel, with a direction opposite to the gravitational field, while the Y-axis is orthogonal to the walls. The channel occupies the region of space $-L \leqslant Y \leqslant L$, the wall at $Y=-L$ is kept at a uniform temperature T_1, while the wall at $Y=L$ is exposed to a uniform heat flux per unit area, q_2, which will be considered as positive if the energy is supplied to the fluid. The dynamic viscosity μ, the thermal expansion coefficient β, the thermal conductivity k and the thermal diffusivity of the fluid will be assumed to be constant. The Boussinesq approximation will be adopted, together with the equation of state for the mass density

$$\rho = \rho_0[1-\beta(T-T_0)] \tag{1}$$

where T_0 is the reference fluid temperature and $\rho_0=\rho(T_0)$. Finally, it will be assumed that the only nonzero component of the velocity field $\mathbf{U}$ is the X-component U. Since $\nabla \cdot \mathbf{U}=0$, U depends only on Y. The momentum balance equation along Y yields

$$\frac{\partial P}{\partial Y} = 0 \tag{2}$$

where $P = p + \rho_0 g X$ is the difference between the pressure and the hydrostatic pressure. Thus, P depends only on X. The momentum balance equation along X yields

$$\rho_0 g \beta (T - T_0) - \frac{dP}{dX} + \mu \frac{d^2 U}{dY^2} = 0. \tag{3}$$

The derivative of Eq. (3) with respect to X yields

$$\frac{\partial T}{\partial X} = \frac{1}{\rho_0 g \beta} \frac{d^2 P}{dX^2}. \tag{4}$$

On account of Eq. (4), $\partial T/\partial X$ is independent of Y. Since $\partial T/\partial X$ is zero for $Y = -L$, T depends only on Y. Thus, Eq. (4) yields

$$\frac{dP}{dX} = \text{constant}. \tag{5}$$

Let us assume that only the X-component of $\mathbf{U}$ is non-zero, the thermal diffusivity is a constant, viscous dissipation is negligible and T depends only on Y. Then, the energy equation reduces to

$$\frac{d^2 T}{dY^2} = 0. \tag{6}$$

Eq. (6) implies that T is a linear function of Y. If $T(L)$ is denoted by T_2, one has

$$T_2 - T_1 = \frac{2Lq_2}{k}, \quad T_2 - T_m = T_m - T_1 = \frac{Lq_2}{k} \tag{7}$$

where the mean fluid temperature T_m, defined as

$$T_m = \frac{1}{2L} \int_{-L}^{L} T(Y)\, dy \tag{8}$$

coincides with the temperature on the plane $Y = 0$. We will call mean fluid density, ρ_m, the density of the fluid at temperature T_m. In analogy with Eq. (8), the mean fluid velocity will be given by

$$U_m = \frac{1}{2L} \int_{-L}^{L} U(Y)\, dY. \tag{9}$$

Let us define the following dimensionless quantities:

$$y = \frac{Y}{L}, \quad u = \frac{U}{U_m}, \quad \theta = \frac{(T - T_m)k}{Lq_2}, \quad Re = \frac{4LU_m\rho_m}{\mu}, \quad Gr = \frac{256\rho_m^2 g\beta q_2 L^4}{\mu^2 k}$$

$$\lambda = -\frac{L^2}{\mu U_m}\frac{dP}{dX}, \quad \gamma = \frac{(T_0 - T_m)k}{Lq_2}, \quad \varepsilon = \beta(T_0 - T_m). \tag{10}$$

The definition of ε and Eq. (1) imply

$$\frac{\rho_0}{\rho_m} = \frac{1}{1+\varepsilon}. \tag{11}$$

Moreover, Eq. (10) yields

$$\frac{Gr}{Re} = \frac{64\rho_m g\beta q_2 L^3}{\mu k U_m}. \tag{12}$$

Contrary to what happens in Refs. [1–3], the Reynolds number Re and the Grashof number Gr do not depend on the reference temperature T_0, either explicitly or through ρ_0. The dimensionless parameters γ and ε have been introduced to analyse the dependence of u and of λ on T_0.

By means of Eq. (10), one can rewrite Eq. (3) in the dimensionless form

$$\frac{d^2 u}{dy^2} = -\lambda - \frac{Gr}{64(1+\varepsilon)\, Re}(\theta - \gamma). \tag{13}$$

Eqs. (9) and (10) yield the following condition on u:

$$\int_{-1}^{1} u(y)\, dy = 2. \tag{14}$$

From Eqs. (6) and (10) one obtains

$$\frac{d^2\theta}{dy^2} = 0. \tag{15}$$

The boundary conditions on θ are

$$\theta(-1) = -1, \quad \left.\frac{d\theta}{dy}\right|_{y=1} = 1. \tag{16}$$

Eqs. (15) and (16) yield

$$\theta(y) = y. \tag{17}$$

By substituting Eq. (17) in Eq. (13) and integrating twice, one obtains

$$u(y) = -\frac{Gr}{384(1+\varepsilon)Re} y^3 + \left(\frac{Gr}{64(1+\varepsilon)Re}\gamma - \lambda\right)\frac{y^2}{2} + Ay + B \tag{18}$$

where A and B are integration constants. The boundary conditions on u,

$$u(-1) = u(1) = 0 \tag{19}$$

yield

$$A = \frac{Gr}{384(1+\varepsilon)Re},$$
$$B = -\frac{1}{2}\left(\frac{Gr}{64(1+\varepsilon)Re}\gamma - \lambda\right). \tag{20}$$

By substituting Eq. (20) in Eq. (18) and applying condition (14), one obtains

$$\lambda = \frac{\gamma}{1+\varepsilon}\,\frac{Gr}{64Re} + 3. \tag{21}$$

Eqs. (18), (20) and (21) yield

$$u(y) = \frac{Gr}{384(1+\varepsilon)Re}y(1-y^2) + \frac{3}{2}(1-y^2). \tag{22}$$

It is easily proved that Eqs. (17), (21) and (22) agree with the results obtained in Refs. [1–3]. By writing γ as $\gamma = \varepsilon k/(\beta L q_2)$ and employing Eq. (12), one can rewrite Eq. (21) in the form

$$\lambda = \frac{\varepsilon}{1+\varepsilon}\Lambda + 3 \tag{23}$$

where

$$\Lambda = \frac{\rho_m g L^2}{\mu U_m} \tag{24}$$

which is preferable in order to study the dependence of λ on the choice of T_0.

In the special case of natural convection, it is convenient to define the modified dimensionless velocity

$$v(y) = \frac{384Re}{Gr}u(y) = \frac{6\mu k}{\rho_m g \beta q_2 L^3}U(Y). \tag{25}$$

From Eqs. (22) and (25) one obtains, for $Re = 0$,

$$v(y) = \frac{1}{1+\varepsilon}y(1-y^2). \tag{26}$$

Eqs. (22) and (26) show that the dimensionless velocities u and v depend on the choice of T_0 through the parameter ε. Reliable values of ε can be obtained from Eq. (10). For liquids at room temperatures, the highest values of β are those of hydrocarbons. For n-pentane at 20°C, $\beta = 1.60 \cdot 10^{-3}$ K^{-1} and, for $|T_0 - T_m| = 30$ K, one has $|\varepsilon| = 0.048 \cong 0.05$. Thus, for liquids in the mixed-convection conditions considered in this paper, one can assume $|\varepsilon| = 0.05$ as an upper bound for $|\varepsilon|$, if T_0 is chosen in the range $T_1 \leqslant T_0 \leqslant T_2$. For ideal gases at 20°C, $\beta = 3.41 \cdot 10^{-3}$ K^{-1} and, for $|T_0 - T_m| = 30$ K, one obtains $|\varepsilon| = 0.102 \cong 0.1$. Thus, $|\varepsilon| = 0.1$ can be considered as a reliable upper bound for $|\varepsilon|$, for the conditions analysed in this paper and for reasonable choices of T_0.

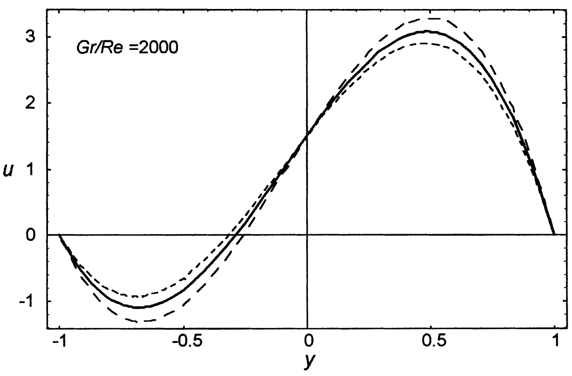

Fig. 1. Boundary condition $T_1 \leftrightarrow q_2$: plots of u vs. y for $Gr/Re = 2000$. The solid line refers to $\varepsilon = 0$, the line with long dashes to $\varepsilon = -0.1$, the line with short dashes to $\varepsilon = 0.1$.

Plots of u vs. y for $\varepsilon = 0$, -0.1 and 0.1, are reported in Fig. 1 for $Gr/Re = 2000$, while plots of v vs. y for natural convection are reported in Fig. 2, for $\varepsilon = 0$, -0.1, and 0.1. In each figure, the solid line refers to $\varepsilon = 0$, the line with long dashes refers to $\varepsilon = -0.1$, the line with short dashes refers to $\varepsilon = 0.1$. Figs. 1 and 2 show that the effect of the choice of T_0 on the dimensionless velocity profiles determined analytically is neither very strong nor negligible. The effect becomes negligible only for low values of Gr/Re.

For mixed convection, the conditions for flow reversal are

$$\left.\frac{du}{dy}\right|_{y=-1} \leq 0 \tag{27}$$

for $Gr/Re > 0$, and

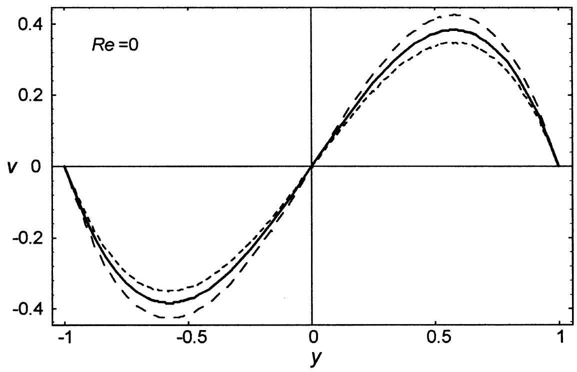

Fig. 2. Boundary condition $T_1 \leftrightarrow q_2$: plots of v vs. y, for natural convection ($Re = 0$). The solid line refers to $\varepsilon = 0$, the line with long dashes to $\varepsilon = -0.1$, the line with short dashes to $\varepsilon = 0.1$

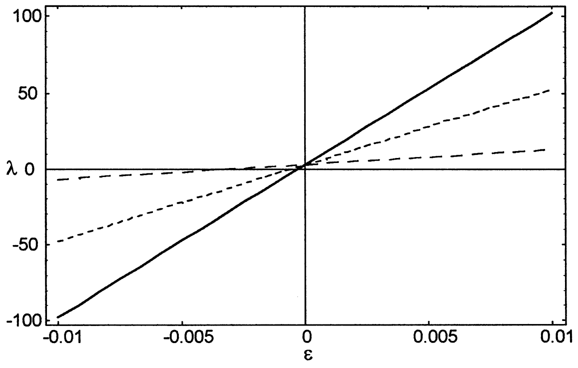

Fig. 3. Boundary condition $T_1 \leftrightarrow q_2$: plots of λ vs. ε. The line with long dashes refers to $\Lambda = 10^3$, the line with short dashes to $\Lambda = 5 \cdot 10^3$, the solid line to $\Lambda = 10^4$.

$$\left.\frac{\mathrm{d}u}{\mathrm{d}y}\right|_{y=1} \leq 0 \tag{28}$$

for $Gr/Re < 0$. Eq. (22) shows that the values of Gr/Re which fulfil conditions (27) and (28) depend on ε. For $\varepsilon = 0$, i.e. $T_0 = T_m$, Eqs. (22) and (27) yield $Gr/Re \geqslant 576$, while Eqs. (22) and (28) yield $Gr/Re \leqslant -576$.

Eq. (23) shows that λ depends on the choice of T_0 through the parameter ε. We will first consider positive values of Λ, which correspond to positive values of U_m (upward motion). Plots of λ vs. ε, for $\Lambda = 10^3$, $5 \cdot 10^3$ and 10^4, are reported in Fig. 3. The line with long dashes refers to $\Lambda = 10^3$, the line with short dashes refers to $\Lambda = 5 \cdot 10^3$, while the solid line refers to $\Lambda = 10^4$. Fig. 3, which represents only a narrow range of values of ε, points out that the effect of the choice of T_0 on λ is very strong, and increases when Λ increases. For water at 20°C, $\rho_m/\mu \cong 10^6$ s m^{-2} and the condition $\Lambda = 10^4$ is obtained, for instance, with $L = 0.01$ m and $U_m = 0.1$ m s^{-1}. For $\varepsilon = 0$, i.e. $T_0 = T_m$, $\lambda = 3$. It is easily verified that this value of λ equals that obtainable for laminar forced convection in a channel. On the other hand, Eq. (23) and Fig. 3 show that, for every choice of T_0 such that $T_0 \neq T_m$, λ becomes negative for sufficiently high values of $|\Lambda|$. Note that $|\Lambda|$ has no upper bound, because it tends to infinity when $|U_m| \to 0$. Indeed, since $1 + \varepsilon > 0$, if Λ is positive (upward flow), negative values of λ may occur for $\varepsilon < 0$, i.e. $T_0 < T_m$. On the other hand, if Λ is negative (downward flow), negative values of λ may occur for $\varepsilon > 0$, i.e. $T_0 > T_m$. The condition $\lambda < 0$ implies that $\mathrm{d}P/\mathrm{d}X$ has the same sign as U_m, i.e. P increases along the motion. This unpleasant result suggests that $\mathrm{d}P/\mathrm{d}X$ can be endowed with an acceptable physical meaning only if the choice $T_0 = T_m$ is made.

Let us now consider the quantity $\mathrm{d}p/\mathrm{d}X$, which is related to $\mathrm{d}P/\mathrm{d}X$ by the equation

$$\frac{\mathrm{d}p}{\mathrm{d}X} = \frac{\mathrm{d}P}{\mathrm{d}X} - \rho_0 g \tag{29}$$

and the dimensionless coefficient

$$\tilde{\lambda} = -\frac{L^2}{\mu U_m}\frac{\mathrm{d}p}{\mathrm{d}X}. \tag{30}$$

From Eqs. (29), (30), (10) and (24) one obtains

$$\tilde{\lambda} = \lambda + \frac{\Lambda}{1+\varepsilon}. \tag{31}$$

Eqs. (31) and (23) yield

$$\tilde{\lambda} = \Lambda + 3. \tag{32}$$

Eq. (32) shows that $\tilde{\lambda}$ is independent of the choice of T_0. Moreover, it shows that, for fixed values of μ, ρ_m and L, $\tilde{\lambda}$ depends only on U_m, with the same law as in the case of forced convection. Thus, for a given channel and for the boundary condition considered in this section, the buoyancy forces have no effect on U_m, if $\mathrm{d}p/\mathrm{d}X$, μ and ρ_m are fixed. This result can be easily argued from Eqs. (21) and (23) only if the choice $T_0 = T_m$ is performed, so that Eqs. (21) and (23) reduce to $\lambda = 3$. Therefore, the choice $T_0 = T_m$ appears as recommendable to simplify the analysis of the relation between the mean velocity and the pressure field.

A more general argument in favour of the choice $T_0 = T_m$ is the following. As it has been shown, different choices of T_0 yield different dimensionless velocity distributions. It is reasonable to argue that the most accurate solution is obtained when Eq. (1) yields the most accurate values of $\rho(T)$ in the domain $-L \leqslant Y \leqslant L$. Eq. (1) can be considered as the first-order truncation of the Taylor series expansion

$$\rho(T) = \rho(T_0) + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{\mathrm{d}^n\rho}{\mathrm{d}T^n}\right)_{T=T_0}(T - T_0)^n. \tag{33}$$

The best accuracy of the first-order truncation of Eq. (33) is obtained by choosing the value of T_0 which minimizes the integral of the squared difference $(T - T_0)^2$ in the domain $-L \leqslant Y \leqslant L$, i.e. the function

$$\mathrm{F}(T_0) = \int_{-L}^{L} (T - T_0)^2 \, \mathrm{d}Y. \tag{34}$$

The minimum of $\mathrm{F}(T_0)$ occurs when its derivative is zero, i.e., when T_0 fulfils the condition

$$\int_{-L}^{L} (T - T_0) \, \mathrm{d}Y = 0. \tag{35}$$

On account of Eq. (8), Eq. (35) is fulfilled if and only if $T_0 = T_m$.

3. Mixed convection problem with the boundary condition $T_1 \leftrightarrow T_2$

Let us consider the boundary condition $T_1 \leftrightarrow T_2$. Eqs (1)–(9) still hold. On account of Eq. (7), one can replace Lq_2/k with $T_2 - T_m$ in the definitions of θ, Gr, and γ, so that these dimensionless parameters can be rewritten as

$$\theta = \frac{T - T_m}{T_2 - T_m}, \quad Gr = \frac{256\rho_m^2 g\beta(T_2 - T_m)L^3}{\mu^2}, \quad \gamma = \frac{T_0 - T_m}{T_2 - T_m}. \tag{36}$$

Thus, Eq. (12) can be written in the form

$$\frac{Gr}{Re} = \frac{64\rho_m g\beta(T_2 - T_m)}{\mu U_m}. \tag{37}$$

Eq. (13) still holds, with the expressions of θ, γ and Gr/Re given above. Eqs. (14) and (15) remain unchanged. On account of Eq. (7), the boundary conditions on θ are $\theta(-1) = -1$ and $\theta(1) = 1$, and yield again Eq. (17). Since the boundary conditions on u are unchanged, one still obtains Eqs. (21)–(32).

4. Solution of the mixed convection problem for the boundary condition $q_1 \leftrightarrow q_2$

Let us consider again the steady and fully developed laminar flow of a Newtonian fluid with constant values of μ, β, k and c_p in the parallel-plate vertical channel described in Section 2. Let us assume that the wall at $Y = -L$ is exposed to a uniform heat flux per unit area q_1, while the wall at $Y = L$ is exposed to a uniform heat flux per unit area q_2. Each heat flux will be considered as positive if the energy is supplied to the fluid. Except in the particular case $q_1 + q_2 = 0$, which reduces to the boundary conditions studied in Sections 2 and 3, the fluid temperature depends on both X and Y. As in Sections 2 and 3, it will be assumed that only the X-component U of the velocity field $\mathbf{U}$ is nonzero. Thus, $\nabla \cdot \mathbf{U} = 0$ implies that U depends only on Y. Moreover, in agreement with Refs. [2,3,7], it will be assumed that $\partial^2 T/\partial X^2 = 0$.

Two different kinds of choice of the reference fluid temperature have been employed in the literature in this case. In Refs. [2] and [3], a constant (and unspecified) reference temperature T_0 has been employed. However, this kind of choice does not appear as suitable for the study of the fully developed region. For instance, let us assume that q_1, q_2 and U_m are positive. In this case, the definitions of the dimensionless quantities employed in Ref. [3], together with Eq. (7) of the same reference, imply that dp/dX is a linearly increasing function of X. Thus, there exists a value X_0 of X such that, for $X > X_0$, p increases along the fluid-flow direction. On the other hand, since the flow is upward, the pressure is expected to be a decreasing function of X. In the study of the fully developed laminar convection for heated vertical pipes, Morton [7] chooses a reference fluid temperature which varies along X, namely the local wall temperature. We will choose as the reference fluid temperature, for each cross section, the mean fluid temperature in the section, i.e.

$$T_m(X) = \frac{1}{2L}\int_{-L}^{L} T(X,Y)\,dY. \tag{38}$$

As a consequence, Eq. (1) will be rewritten as

$$\rho = \rho_m[1 - \beta(T - T_m)] \tag{39}$$

where $\rho_m = \rho(T_m)$. The momentum balance equation along Y yields $\partial P/\partial Y = 0$, so that $P = P(X)$. The momentum balance equation along X yields

$$\rho_m g\beta(T - T_m) - \frac{dP}{dX} + \mu\frac{d^2U}{dY^2} = 0. \tag{40}$$

Clearly, since T_m depends on X, ρ_m depends on X as well. However, in analogy with Ref. [7], we will assume that the derivative $d\rho_m/dX$ is negligible. With this assumption, the derivative of Eq. (40) with respect to X yields

$$\frac{\partial T}{\partial X} = \frac{dT_m}{dX} + \frac{1}{\rho_m g\beta}\frac{d^2P}{dX^2}. \tag{41}$$

Since T_m and P depend only on X, Eq. (41) implies that $\partial T/\partial X$ is independent of Y. Thus, one obtains

$$\frac{\partial T}{\partial X} = \frac{dT_m}{dX}. \tag{42}$$

Eqs. (41) and (42) yield $d^2P/dX^2 = 0$, i.e.

$$\frac{dP}{dX} = \text{constant}. \tag{43}$$

The energy balance equation can be written as

$$\frac{\partial T}{\partial X}U = \frac{k}{\rho_m c_p}\frac{\partial^2 T}{\partial Y^2}. \tag{44}$$

Since Eq. (44) is non-linear, the method to determine the reference temperature proposed in Ref. [6] is ruled out. Eqs. (42) and (44) yield

$$\rho_m c_p \frac{dT_m}{dX} U = k \frac{\partial^2 T}{\partial Y^2}. \tag{45}$$

An integration of Eq. (45) from $-L$ to L gives

$$2L\rho_m c_p U_m \frac{dT_m}{dX} = k \frac{\partial T}{\partial Y}\bigg|_{Y=L} - k \frac{\partial T}{\partial Y}\bigg|_{Y=-L}. \tag{46}$$

The thermal boundary conditions are

$$-k \frac{\partial T}{\partial Y}\bigg|_{Y=-L} = q_1, \quad k \frac{\partial T}{\partial Y}\bigg|_{Y=L} = q_2. \tag{47}$$

Eqs. (46) and (47) yield

$$\frac{dT_m}{dX} = \frac{q_1 + q_2}{2L\rho_m c_p U_m}. \tag{48}$$

From Eqs. (45) and (48) one obtains

$$\frac{\partial^2 T}{\partial Y^2} = \frac{q_1 + q_2}{2Lk} \frac{U}{U_m}. \tag{49}$$

Let us introduce the dimensionless quantities y, u, θ, Re, Gr and λ defined in Eq. (10), and the dimensionless parameter

$$R = \frac{q_1}{q_2}. \tag{50}$$

Obviously, Gr/Re is still given by Eq. (12). Moreover, an account of Eq. (42), θ depends only on y. Eq. (40) can be rewritten as

$$\frac{d^2u}{dy^2} = -\frac{Gr}{64Re}\theta - \lambda \tag{51}$$

while Eq. (49) yields

$$\frac{d^2\theta}{dy^2} = \frac{1+R}{2}u. \tag{52}$$

The boundary conditions on u are given by Eq. (19). From Eq. (47) one obtains

$$\frac{d\theta}{dy}\bigg|_{y=-1} = -R, \quad \frac{d\theta}{dy}\bigg|_{y=1} = 1. \tag{53}$$

Moreover, Eq. (14) holds, together with the condition

$$\int_{-1}^{1} \theta(y)\, dy = 0. \tag{54}$$

Eqs. (51) and (52) yield

$$\frac{d^4u}{dy^4} = -\frac{Gr}{128Re}(1+R)u. \tag{55}$$

From Eqs. (51) and (53) one obtains

$$\frac{d^3u}{dy^3}\bigg|_{y=-1} = \frac{R\,Gr}{64Re}, \quad \frac{d^3u}{dy^3}\bigg|_{y=1} = -\frac{Gr}{64Re}. \tag{56}$$

Eq. (51) can be rewritten as

$$\theta = -\frac{64Re}{Gr}\left(\frac{d^2u}{dy^2} + \lambda\right). \tag{57}$$

Eqs. (54) and (57) yield

$$\lambda = \frac{1}{2}\left(\frac{du}{dy}\bigg|_{y=-1} - \frac{du}{dy}\bigg|_{y=1}\right). \tag{58}$$

Let us now solve Eq. (55), together with the boundary conditions (19) and (56) and with the constraint given by Eq. (14). If $(Gr/Re)(1+R) \neq 0$, the general solution of Eq. (55) can be determined by solving the characteristic equation

$$\eta^4 = \omega^4 \tag{59}$$

where

$$\omega = \frac{1}{2}\left(-\frac{1+R}{8}\frac{Gr}{Re}\right)^{1/4}. \tag{60}$$

Since the solutions of Eq. (59) are ω, $-\omega$, $i\omega$ and $-i\omega$, the general solution of Eq. (55) can be written in the form

$$u(y) = a_1 \sinh(\omega y) + a_2 \cosh(\omega y) + a_3 \sin(\omega y) + a_4 \cos(\omega y). \tag{61}$$

The coefficients a_1, a_2, a_3 and a_4 are determined by the boundary conditions (19) and (56). By substituting the expressions of these coefficients in Eq. (61), one obtains

$$u(y) = \frac{(1-R)\omega[\sin\omega\ \sinh(\omega y) - \sinh\omega\ \sin(\omega y)]}{(1+R)(\sin\omega\ \cosh\omega + \cos\omega\ \sinh\omega)} + \frac{\omega[\cosh\omega\ \cos(\omega y) - \cos\omega\ \cosh(\omega y)]}{(\sin\omega\ \cosh\omega - \cos\omega\ \sinh\omega)}. \tag{62}$$

Clearly, the first term on the right-hand-side of Eq. (62) vanishes for $R = 1$.

Eqs. (58) and (62) yield

$$\lambda = \frac{\omega^2(\sin\omega\ \cosh\omega + \cos\omega\ \sinh\omega)}{\sin\omega\ \cosh\omega - \cos\omega\ \sinh\omega}. \tag{63}$$

Then, from Eqs. (57), (60), (62) and (63) one obtains

$$\theta(y) = \frac{1+R}{2\omega^2}\left\{\frac{(1-R)\omega[\sin\omega\ \sinh(\omega y) + \sinh\omega\ \sin(\omega y)]}{(1+R)(\sin\omega\ \cosh\omega - \cos\omega\ \sinh\omega)} + \frac{\sin\omega\ \cosh\omega + \cos\omega\ \sinh\omega - \omega[\cos\omega\ \cosh(\omega y) + \cosh\omega\ \cos(\omega y)]}{\sin\omega\ \cosh\omega - \cos\omega\ \sinh\omega}\right\}. \tag{64}$$

Eqs. (62), (63) and (64) hold for $\omega \neq 0$ and $R \neq -1$. The special cases $\omega = 0$ and $R = -1$ will be considered later. The velocity profiles implied by Eq. (62) agree with those obtained by Cheng et al. [3], provided that the Grashof number is evaluated at the local mean fluid temperature $T_m(X)$. Both for $R = 1$ and for $R \neq 1$, $u(y)$ and $\theta(y)$ become singular for a sequence of values of ω. For $R \neq 1$, singularities occur for the real values of ω which are roots of the equation

$$(\sin\omega\ \cosh\omega)^2 - (\cos\omega\ \sinh\omega)^2 = 0. \tag{65}$$

The first five real roots of Eq. (65) are $\omega_0 \cong 2.36502037$, $\omega_1 \cong 3.92660231$, $\omega_2 \cong 5.49780391$, $\omega_3 \cong 7.06858274$, $\omega_4 \cong 8.63937982$. Only the second and the fourth of the values of ω reported above correspond to singularities of $u(y)$ and $\theta(y)$ for $R = 1$. However, since the condition $R = 1$ cannot be obtained experimentally with an infinite accuracy, all the values of ω reported above correspond to singularities of $u(y)$ and $\theta(y)$ in practical cases. On account of Eq. (60), all singularities occur for $(1+R)Gr/Re < 0$. Thus, if $q_1 + q_2$ is positive all singularities occur for downward flow, while if $q_1 + q_2$ is negative all singularities occur for upward flow. In particular, ω_0 corresponds to $(1+R)Gr/Re = -128\omega_0^4 \cong -4004.51$. In analogy with Morton [7], we will suppose that fluid flows with $(1+R)Gr/Re < -128\omega_0^4$ cannot be obtained experimentally.

In the special cases $Gr/Re = 0$ and $R = -1$, Eqs. (62)–(64) do not hold. If $Gr/Re = 0$, i.e., if purely forced convection occurs, Eq. (55) reduces to

$$\frac{d^4u}{dy^4} = 0. \tag{66}$$

By integrating Eq. (66), with the boundary conditions (19) and (56) and the constraint (14), one obtains the Hagen–Poiseuille velocity profile

$$u(y) = \tfrac{3}{2}(1 - y^2). \tag{67}$$

Eqs. (58) and (67) yield

$$\lambda = 3. \tag{68}$$

The substitution of Eq. (67) in Eq. (52) gives

$$\frac{d^2\theta}{dy^2} = \frac{3}{4}(1+R)(1-y^2). \tag{69}$$

The integration of Eq. (69), with the boundary conditions (53) and the constraint (54) yields

$$\theta = \frac{3}{4}(1+R)\left(-\frac{y^4}{12} + \frac{y^2}{2} + \frac{2}{3}\frac{1-R}{1+R}y - \frac{9}{60}\right). \tag{70}$$

If $R = -1$, Eq. (55) reduces again to Eq. (66). By integrating Eq. (66), with the boundary conditions (19) and (56) and the constraint (14), one obtains the velocity profile

$$u(y) = \frac{Gr}{384Re}y(1-y^2) + \frac{3}{2}(1-y^2). \tag{71}$$

Note that, in the case $\varepsilon = 0$, Eq. (22) reduces to Eq. (71).

Eqs. (58) and (71) yield Eq. (68), while Eq. (52) reduces to Eq. (15). The integration of Eq. (15), with the boundary conditions (53) and the constraint (54), yields Eq. (17). Indeed, for $R = -1$ the boundary condition $q_1 \leftrightarrow q_2$ coincides with the boundary conditions $T_1 \leftrightarrow q_2$ and $T_1 \leftrightarrow T_2$.

Let us now discuss the implications of Eqs. (62)–(64). The unlikely previsions of Eqs. (62) and (64) for $\omega \cong \omega_0$ and $R \cong 1$ are illustrated in Fig. 4, which represents plots of u and θ vs. y for $(1+R)Gr/Re = -4004.51121392346$ and $R = 0.998$, 0.999, 1.001, 1.002. The figure shows that, although the boundary conditions are very close to symmetry, the plots of u and θ seem perfectly antisymmetric. Positive values of u and negative values of θ are predicted for $y < 0$ if $R < 1$, and for $y > 0$ if $R > 1$. The absolute values of u and θ exceed, respectively, $6 \cdot 10^{12}$ and $2 \cdot 10^{12}$ for both $R = 0.998$ and 1.002; they exceed, respectively,

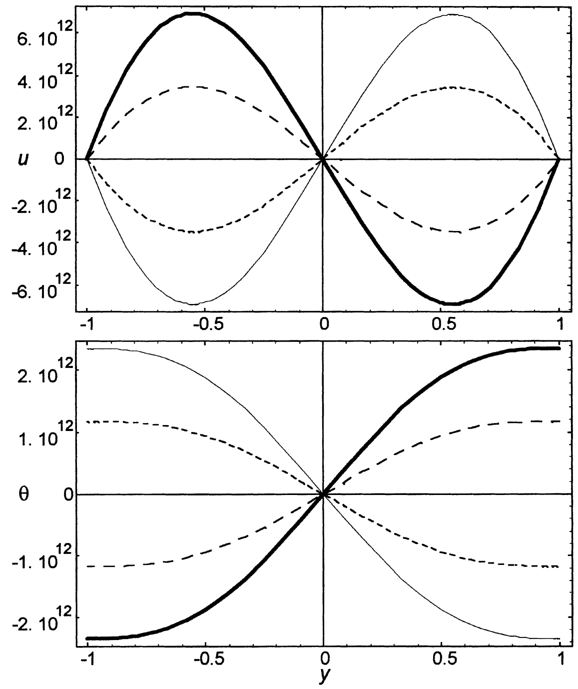

Fig. 4. Boundary condition $q_1 \leftrightarrow q_2$: plots of u and θ vs. y for $(1+R)Gr/Re = -4004.51121392346$ and values of R close to $R=1$. The thick solid lines refer to $R=0.998$, the lines with long dashes to $R=0.999$, the lines with short dashes to $R=1.001$, the thin solid lines to $R=1.002$.

$3 \cdot 10^{12}$ and $1 \cdot 10^{12}$ for both $R=0.999$ and 1.001. Obviously, the predictions plotted in Fig. 4 cannot correspond to physical reality. Indeed, this figure illustrates, by examples, the following circumstance. For every value of R such that Eqs. (62)–(64) hold, except if R is exactly equal to 1, the predictions of these equations are quite unlikely for $Gr/Re = -128\omega_0^4/(1+R)$. This circumstance suggests that the laminar fluid flows considered in this section can be attained experimentally only if $|Gr/Re| < 128\omega_0^4/(1+R)$. In particular, for $R=1$ this condition implies $|Gr/Re| <$ 2002.26.

In recent theoretical works on the stability of mixed-convection flow in vertical channels, Chen and Chung [8,9] find more restrictive stability conditions. For symmetrically heated channels [8], they conclude that, for $Pr=7$ and $|Re|>40$, buoyancy assisted flow (i.e., for instance, upward flow with $q_2=q_1>0$) can become unstable if $Gr/Re>1054$ and buoyancy opposed flow can become unstable if $Gr/Re < -390$. On the other hand, for asymmetrically heated channels ($R=-1$), the flow stability is strongly dependent on Re and Pr [9]. Clearly, no upper limit on $|Gr/Re|$ can be established in this case if Re is not fixed, because, if $R=-1$, no net heat flux is supplied to the fluid and natural convection ($Re=0$) can occur. The available experimental studies on the stability of mixed-convection flow do not refer to vertical channels, but to vertical pipes [10]. The authors [10] conclude that, for uniformly heated pipes, buoyancy assisted flow becomes unstable when the velocity profiles develop points of inflection. For a symmetrically heated vertical channel, the velocity profiles develop points of inflection for $Gr/Re \geqslant 1558.55$. To the authors' knowledge, neither theoretical nor experimental works provide stability criteria for the mixed-convection flow in a vertical channel with $|R| \neq 1$.

In the following, we will consider $q_2>0$ and $-1 \leqslant R \leqslant 1$. We will assume the conditions $Gr/Re \leqslant 1000$ for upward flow and $Gr/Re \geqslant -350$ for downward flow.

Fig. 5 illustrates the dependence of the dimensionless-velocity profiles on the value of R, for upward

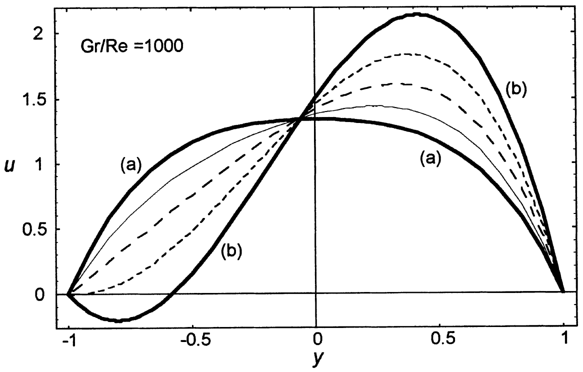

Fig. 5. Boundary condition $q_1 \leftrightarrow q_2$: plots of u vs. y for $Gr/Re=1000$ and some values of R. The thick solid line (a) refers to $R=1$, the thin solid line to $R=0.5$, the line with long dashes to $R=0$, the line with short dashes to $R=-0.5$, the thick solid line (b) to $R=-1$.

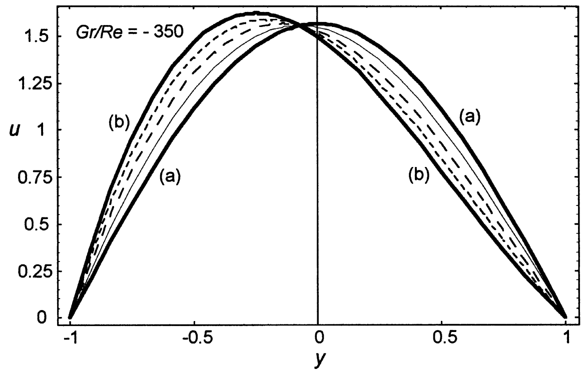

Fig. 6. Boundary condition $q_1 \leftrightarrow q_2$: plots of u vs. y for $Gr/Re=-350$ and some values of R. The thick solid line (a) refers to $R=1$, the thin solid line to $R=0.5$, the line with long dashes to $R=0$, the line with short dashes to $R=-0.5$, the thick solid line (b) to $R=-1$.

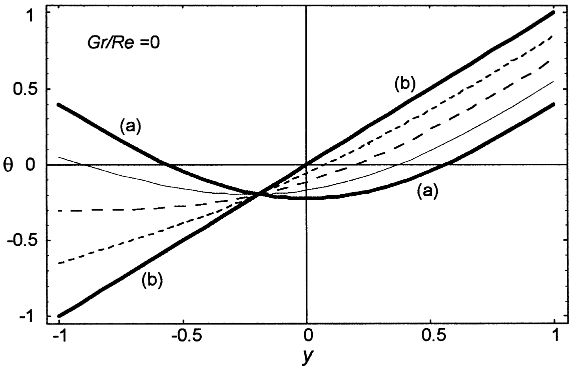

Fig. 7. Boundary condition $q_1 \leftrightarrow q_2$: plots of θ vs. y for $Gr/Re=0$ and some values of R. The thick solid line (a) refers to $R=1$, the thin solid line to $R=0.5$, the line with long dashes to $R=0$, the line with short dashes to $R=-0.5$, the thick solid line (b) to $R=-1$.

flow. The figure shows that, for $Gr/Re=1000$, the plots of u vs. y undergo sharp changes while R varies in the range $-1 \leqslant R \leqslant 1$. The thick solid line denoted by (a), which represents the condition $R=1$, displays a velocity profile similar to that of forced convection, but with $u(0)\cong 1.335$. No inflection of the velocity profile is present. On the other hand, the thick solid line denoted by (b), which represents the condition $R=-1$, shows an appreciable flow reversal close to the cooled wall. The plots for $R=0.5$ (thin solid line), $R=0$ (line with long dashes), and $R=-0.5$ (line with short dashes) lie between the lines denoted by (a) and (b); clearly, the plot for $R=0.5$ is closer to line (a), while that for $R=-0.5$ is closer to line (b). A hardly appreciable flow reversal occurs for $R=-0.5$. The dependence of $u(y)$ on R for downward flow is illustrated in Fig. 6, which refers to $Gr/Re=-350$ with the same values of R as Fig. 5. The influence of R on the velocity profiles is still appreciable, even if the absolute value of Gr/Re is rather small. Clearly, for $R \neq 1$, the highest values of u occur for $y < 0$, i.e. in the neighbourhood of the cooled wall.

The plots of the dimensionless temperature profiles corresponding to the plots of $u(y)$ represented in Figs. 5 and 6 have not been reported, because the dependence of $\theta(y)$ on Gr/Re is hardly appreciable in graphic form. Plots of $\theta(y)$ for $Gr/Re=0$ (forced convection) and some values of R are reported in Fig. 7. As expected, the plots for $R=0.5$ (thin solid line), $R=0$ (line with long dashes), and $R=-0.5$ (line with short dashes) lie between the solid lines denoted by (a) and (b), which refer to $R=1$ and to $R=-1$, respectively.

Plots of λ vs. Gr/Re, in the range $-350 \leqslant Gr/Re \leqslant 1000$, are reported in Fig. 8 for $R=1$ [line (a)], $R=0.5$, 0, -0.5 and -1 [line (b)]. The figure shows that, while for $R=-1$ the parameter λ is constant, for $R>-1$ it is an increasing function of Gr/Re. Thus, if $R>-1$ and the value of $\mu U_m/L^2$ is fixed, the absolute value of dP/dX is higher for buoyancy-assisted flow than for buoyancy-opposed flow. This effect is due merely to the change of shape of the velocity profile. In fact, the choice of $T_m(X)$ as the reference fluid temperature implies that the average value of the buoyancy force in each channel section is zero. As a consequence, strictly speaking, the buoyancy neither assists nor opposes the net fluid flow. In this paper, the terms buoyancy-assisted flow and buoyancy-opposed flow are used only to denote the cases $Gr/Re>0$ and $Gr/Re<0$.

The Nusselt numbers at the channel walls will be defined as

$$Nu_1 = \frac{4Lq_1}{k(T_1 - T_b)}, \quad Nu_2 = \frac{4Lq_2}{k(T_2 - T_b)} \tag{72}$$

where Nu_1 refers to the wall at $Y=-L$, Nu_2 refers to the wall at $Y=L$, and

$$T_b = \frac{1}{2LU_m}\int_{-L}^{L} UT\,dY \tag{73}$$

is the bulk temperature of the fluid in the cross section considered. By employing Eqs. (10) and (50), Eq. (72) can be rewritten as

$$Nu_1 = \frac{4R}{\theta_1 - \theta_b}, \quad Nu_2 = \frac{4}{\theta_2 - \theta_b} \tag{74}$$

where the dimensionless bulk temperature θ_b is given by

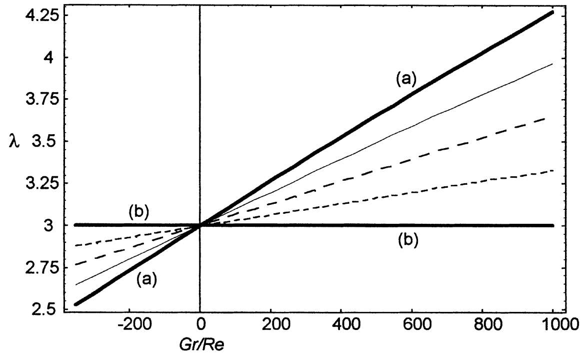

Fig. 8. Boundary condition $q_1 \leftrightarrow q_2$: plots of λ vs. Gr/Re for some values of R. The thick solid line (a) refers to $R=1$, the thin solid line to $R=0.5$, the line with long dashes to $R=0$, the line with short dashes to $R=-0.5$, the thick solid line (b) to $R=-1$.

Table 1
Values of the Nusselt numbers at the channel walls, for the boundary condition $q_1 \leftrightarrow q_2$

	$R=1$		$R=0.5$		$R=0$		$R=-0.5$		$R=-1$	
Gr/Re	Nu_1	Nu_2	Nu_1	Nu_2	Nu_1	Nu_2	Nu_1	Nu_2	Nu_1	Nu_2
1000	8.8200	8.8200	15.187	7.4526	0.0000	6.5674	2.7494	6.0910	2.9691	6.1277
800	8.7053	8.7053	15.492	7.2888	0.0000	6.3521	2.8080	5.7723	3.1304	5.5385
600	8.5894	8.5894	15.861	7.1148	0.0000	6.1266	2.8775	5.4627	3.3103	5.0526
400	8.4725	8.4725	16.306	6.9286	0.0000	5.8905	2.9605	5.1625	3.5122	4.6452
300	8.4136	8.4136	16.562	6.8305	0.0000	5.7683	3.0081	5.0158	3.6226	4.4651
200	8.3545	8.3545	16.845	6.7284	0.0000	5.6433	3.0604	4.8716	3.7403	4.2985
100	8.2950	8.2950	17.157	6.6223	0.0000	5.5154	3.1181	4.7297	3.8658	4.1439
0	8.2353	8.2353	17.500	6.5116	0.0000	5.3846	3.1818	4.5902	4.0000	4.0000
−100	8.1753	8.1753	17.879	6.3960	0.0000	5.2509	3.2525	4.4530	4.1439	3.8658
−200	8.1150	8.1150	18.296	6.2750	0.0000	5.1142	3.3311	4.3183	4.2985	3.7403
−300	8.0545	8.0545	18.755	6.1481	0.0000	4.9746	3.4189	4.1859	4.4651	3.6226

$$\theta_b = \frac{1}{2} \int_{-1}^{1} u\theta \, dy. \quad (75)$$

The Nusselt numbers defined above can be employed to evaluate $T_1 - T_b$, $T_2 - T_b$ and $T_1 - T_2$, if q_1 and q_2 are given. In particular, $\theta_1 - \theta_2 = 4(R/Nu_1 - 1/Nu_2)$. Values of Nu_1 and of Nu_2 in the range $-300 \leqslant Gr/Re \leqslant 1000$ are reported in Table 1, for $R=1$, 0.5, 0, -0.5 and -1. For positive values of R and of Gr/Re, it is possible to verify that the values of Nu_1 and of Nu_2 reported in Table 1 agree with those evaluated by Cheng et al. [3]. To perform this check, one must consider that the ratio between the Grashof number and the Reynolds number defined in this paper is eight times that defined in Ref. [3], while the Nusselt numbers defined in this paper are twice those defined in Ref. [3].

The results reported in Table 1 show that Nu_1 is an increasing function of Gr/Re for $R=1$, vanishes for $R=0$, and is a decreasing function of Gr/Re for the other values of R. On the other hand, Nu_2 is an increasing function of Gr/Re for each value of R and, obviously, equals Nu_1 for $R=1$.

5. Conclusions

The fully-developed mixed-convection in a plane vertical channel has been studied analytically. For the boundary conditions of either uniform wall temperatures or a uniform wall temperature and a uniform wall heat flux, it has been shown that the choice of the reference fluid temperature has a non-negligible effect on the dimensionless velocity profiles and a very sharp effect on the gradient of the difference P between the pressure and the hydrostatic pressure. In particular, only if the mean fluid temperature T_m is chosen as the reference temperature, P always decreases along the flow direction. A more general argument in favour of the choice of the mean fluid temperature in a cross section T_m as the reference fluid temperature for any fully-developed mixed-convection problem in channels has been proposed. For the boundary condition of uniform wall heat fluxes q_1 and q_2, it has been proved that any choice of a fixed reference temperature yields an unlikely pressure field. Therefore, the mean fluid temperature in each cross section, $T_m(X)$, has been chosen as a local reference temperature. With this choice, an analytical solution of the fully-developed mixed-convection in a vertical channel with the boundary condition of uniform wall heat fluxes has been presented. The solution holds also for negative values of q_1/q_2. In particular, for $q_1/q_2 = -1$ it coincides with the solution obtained for the boundary conditions of uniform wall temperatures and of a uniform wall temperature and a uniform wall heat flux.

References

[1] W. Aung, G. Worku, Theory of fully developed, combined convection including flow reversal, ASME Journal of Heat Transfer 108 (1986) 485–488.

[2] T.T. Hamadah, R.A. Wirtz, Analysis of laminar fully developed mixed convection in a vertical channel with opposing buoyancy, ASME Journal of Heat Transfer 113 (1991) 507–510.

[3] C.-H. Cheng, H.-S. Kou, W.-H. Huang, Flow reversal and heat transfer of fully developed mixed convection in vertical channels, Journal of Thermophysics and Heat Transfer 4 (1990) 375–383.

[4] A. Barletta, Laminar mixed convection with viscous dissipation in a vertical channel, International Journal of Heat and Mass Transfer 41 (1998) 3501–3513.

[5] E. Zanchini, Effect of viscous dissipation on mixed convection in a vertical channel with boundary conditions

of the third kind, International Journal of Heat and Mass Transfer 41 (1998) 3949–3959.
[6] V.S. Arpaci, P.S. Larsen, Convection heat transfer, Prentice-Hall, Englewood Cliffs, NJ, 1984 Chapter 3.
[7] B.R. Morton, Laminar convection in uniformly heated vertical pipes, Journal of Fluid Mechanics 8 (1960) 227–240.
[8] Yen-Cho Chen, J.N. Chung, The linear stability of mixed convection in a vertical channel flow, Journal of Fluid Mechanics 325 (1996) 29–51.
[9] Yen-Cho Chen, J.N. Chung, Stability of mixed convection in a differentially heated vertical channel, ASME Journal of Heat Transfer 120 (1998) 127–132.
[10] G.F. Scheele, T.J. Hanratty, Effect of natural convection on transition to turbulence in vertical pipes, Journal of Fluid Mechanics 14 (1962) 244–256.